Integrál-transzformációk, Jacobi determináns

Síkbeli polárkoordináták

$$\left.\begin{matrix} x = \varrho\,\cos\varphi \\ y = \varrho\,\sin\varphi \end{matrix}\right\} \left\{\begin{matrix} 0 \le r \textit{ adott} \\ 0 \le \varrho \le r \\ 0 \le \varrho < 2\pi \end{matrix}\right\} \quad \Rightarrow \quad \frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{pmatrix} \frac{\partial x}{\partial \varrho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \varrho} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\varrho\sin\varphi \\ \sin\varphi & \varrho\cos\varphi \end{pmatrix} \quad \Rightarrow \quad \left|\frac{\partial(x,y)}{\partial(r,\varphi)}\right| = \varrho$$

Henger-koordináták

$$\left.\begin{matrix} x = \varrho\,\cos\varphi \\ y = \varrho\,\sin\varphi \\ z = z \end{matrix}\right\} \quad \left\{\begin{matrix} 0 \le r \textit{ adott} \\ 0 \le \varrho \le r \\ 0 \le \varphi \le 2\pi \end{matrix}\right\} \quad \Rightarrow \quad \frac{\partial(x,y,z)}{\partial(\varrho,\varphi,z)} = \begin{pmatrix} \frac{\partial x}{\partial \varrho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \varrho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \varrho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \varrho\sin\varphi & 0 \\ \sin\varphi & -\varrho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow$$

$$\Rightarrow \quad \left|\frac{\partial(x,y,z)}{\partial(\varrho,\varphi,z)}\right| = \varrho$$

Gömbi (szférikus) koordináták

$$\left.\begin{matrix} x = \varrho\,\cos\varphi\,\sin\vartheta \\ y = \varrho\,\sin\varphi\,\sin\vartheta \\ z = \varrho\,\cos\vartheta \end{matrix}\right\} \quad \left\{\begin{matrix} 0 < r \textit{ adott} \\ 0 \le \varphi \le 2\pi \\ 0 \le \vartheta \le \pi \end{matrix}\right\} \quad \Rightarrow \quad \frac{\partial(x,y,z)}{\partial(\varrho,\varphi,\vartheta)} = \begin{pmatrix} \frac{\partial x}{\partial \varrho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \vartheta} \\ \frac{\partial y}{\partial \varrho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \vartheta} \\ \frac{\partial z}{\partial \varrho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \vartheta} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\varphi\,\sin\vartheta & -\varrho\,\sin\varphi\,\sin\vartheta & \varrho\,\cos\varphi\,\cos\vartheta \\ \sin\varphi\,\sin\vartheta & \varrho\,\cos\varphi\,\sin\vartheta & \varrho\,\sin\varphi\,\cos\vartheta \\ \cos\vartheta & 0 & -\varrho\,\sin\vartheta \end{pmatrix} \quad \Rightarrow \quad \left|\frac{\partial(x,y,z)}{\partial(\varrho,\varphi\vartheta)}\right| = \varrho^2\sin\vartheta$$

Tórusz koordináták

$$\left.\begin{array}{l} x = (R+\varrho\cos\psi)\cos\varphi \\ y = (R+\varrho\cos\psi)\sin\varphi \\ z = \qquad\qquad \varrho\sin\psi \end{array}\right\} \qquad \left\{\begin{array}{l} 0 \le r \le R \ adottak \\ 0 \le \varphi,\psi < 2\pi \\ 0 \le \varrho \le r \end{array}\right\}$$

$$\frac{\partial(x,y,z)}{\partial(\varrho,\varphi,\psi)} = \begin{pmatrix} \frac{\partial x}{\partial\varrho} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\psi} \\ \frac{\partial y}{\partial\varrho} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\psi} \\ \frac{\partial z}{\partial\varrho} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\psi} \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\psi & -(R+\varrho\cos\psi)\sin\varphi & -\varrho\sin\psi\cos\varphi \\ \sin\varphi\cos\psi & (R+\varrho\cos\psi)\cos\varphi & -\varrho\sin\psi\sin\varphi \\ \sin\psi & 0 & \varrho\cos\psi \end{pmatrix} \Rightarrow$$

$$\left|\frac{\partial(x,y,z)}{\partial(\varrho,\varphi,\psi)}\right| = \varrho(R+\varrho\cos\psi)$$

Tórusz felszíne

Tórusz, mint kétparaméteres (φ,ψ) felület. $(\varrho = r)$

$$\boldsymbol{r}'_\psi = \begin{pmatrix} -r\sin\psi\cos\varphi \\ -r\sin\psi\sin\varphi \\ r\cos\psi \end{pmatrix} = -r\begin{pmatrix} \sin\psi\cos\varphi \\ \sin\psi\sin\varphi \\ -\cos\psi \end{pmatrix}$$

$$\boldsymbol{r}'_\varphi = \begin{pmatrix} -(R+r\cos\psi)\sin\varphi \\ (R+r\cos\psi)\cos\varphi \\ 0 \end{pmatrix} = (R+r\cos\psi)\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$\boldsymbol{r}'_\psi \times \boldsymbol{r}'_\varphi = -r(R+r\cos\psi)\begin{pmatrix} \cos\psi\cos\varphi \\ \cos\psi\sin\varphi \\ \sin\psi \end{pmatrix} \Rightarrow \left|\boldsymbol{r}'_\psi \times \boldsymbol{r}'_\varphi\right| = r(R+r\cos\psi)$$

Kétparaméteres felület $\boldsymbol{R}^2 \to \boldsymbol{R}^3$

$$\boldsymbol{r} = x(u,v)\cdot\boldsymbol{i} + y(u,v)\cdot\boldsymbol{j} + z(u,v)\cdot\boldsymbol{k}$$

Érintősík normálvektora és felszíne:

$$\boldsymbol{n} = \boldsymbol{r}_u' \times \boldsymbol{r}_v' \qquad A = \iint_{\mathcal{T}} \left|\boldsymbol{r}_u' \times \boldsymbol{r}_v{}'\right| du\,dv$$

$z = f(x,y)$ típusú előállítás esetén:

$$\boldsymbol{n} = -f_x'(x,y)\boldsymbol{i} - f_y'(x,y)\boldsymbol{j} + 1\boldsymbol{k} \qquad A = \iint_{\mathcal{T}} \sqrt{f_x'^2(x,y) + f_y'^2(x,y) + 1}\, du\,dv$$

Kettős integrál fizikai alkalmazásai

Egy $\mathcal{T}$ síklemez tartományon a lemez sűrűségét a
$\varrho(x,y) \quad ((x,y) \in \mathcal{T})$ függvény írja le.
A tömegközéppont koordinátái:

$$x_s = \frac{\iint_{\mathcal{T}} x\varrho(x,y)\,dx\,dy}{\iint_{\mathcal{T}} \varrho(x,y)\,dx\,dy}\,; \qquad y_s = \frac{\iint_{\mathcal{T}} y\varrho(x,y)\,dx\,dy}{\iint_{\mathcal{T}} \varrho(x,y)\,dx\,dy}$$

(Számláló: a síklemez y ill. x tengelyre vett elsőrendű, *(statikai) nyomatéka.*
Nevező: a lemez tömege.)

Az y ill. x tengelyre vett másodrendű, (tehetetlenségi) nyomatéka:

$$\Theta_x = \iint_{\mathcal{T}} y^2\varrho(x,y)\,dx\,dy\,; \qquad \Theta_y = \iint_{\mathcal{T}} x^2\varrho(x,y)\,dx\,dy$$

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matek.x3.hu

matek@x3.hu